Surfaces whose Lines of Curvature in one System are Represented on the Sphere by Great Circles.

By L. P. Eisenhart.

Guichard* has shown that the determination of all congruences whose developables have a given spherical representation, reduces to the solution of an equation of Laplace, after the direction cosines have been found. We apply this method to the case where one family of lines on the sphere is composed of great circles and the other consists of their orthogonal trajectories. It is evident that all of these congruences are normal, and that the lines on the sphere are the images of the lines of curvature of the parallel surfaces which cut the lines orthogonally. This furnishes a means for the determination and study of surfaces whose lines of curvature in one system are represented on the sphere by great circles.

In the first place, we find that when such a configuration is given upon the sphere, the determination of the direction cosines of the lines of the congruence is the same problem as the finding of a skew curve from its intrinsic equations. The equation of Laplace for this special case can be solved by two quadratures, and as two arbitrary functions are introduced, there is a double infinity of families of parallel surfaces whose lines of curvature have the given representation.

It is found that these surfaces are characterized by the property that one family of the lines of curvature are geodesics, and along the lines of curvature of the second system one of the principal radii is constant. From these properties it follows that one of the sheets of the evolute is a developable surface, and

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* "Surfaces rapportées à leurs lignes asymptotiques et congruences rapportées à leurs developpables" (Annales Scientifiques de l'École Normale Supérieure, t. VI, 3e serie).

conversely. Hence, the surfaces under consideration are the *surfaces of Monge*,* and consequently can be generated by a plane curve whose plane rolls without sliding upon a developable surface. It is shown that one of the arbitrary functions, which appear in the expression for the semi-focal distance, depends entirely upon the generating curve, and that the other function in connection with the spherical representation determines the character of the generating developable. In particular, the *moulure* surfaces are considered from this point of view.

The second sheet of the evolute is a surface of Monge with the same surface generator, and its generating curve is the evolute of the curve for the surface. These curves are represented on the sphere by the same great circle and corresponding points are at the distance of a quadrant.

Surfaces of revolution form a subclass of moulure surfaces, corresponding to the case where the surface generator is a straight line. It is evident that there are only particular families of great circles which can be the images of the meridians. For these systems the direction cosines can be found by quadratures, and hence the complete determination of all surfaces of revolution, whose meridians have a given representation, reduces to quadratures.

Finally, we show that surfaces of revolution are the only surfaces of Monge which are Weingarten surfaces, and that they are the only isothermic surfaces of Monge.

1. Consider a sphere of radius unity and with center at the origin of coordinates, and let X, Y, Z denote the cartesian coordinates of a point on the sphere, or, what is the same thing, the direction cosines of the radius. Let the sphere be referred to a system of lines $v =$ const., $u =$ const., and write

$$\mathcal{E} = \Sigma \left(\frac{\partial X}{\partial u}\right)^2, \quad \mathcal{F} = \Sigma \frac{\partial X}{\partial u} \frac{\partial X}{\partial v}, \quad \mathcal{G} = \Sigma \left(\frac{\partial X}{\partial v}\right)^2. \tag{1}$$

In particular, we consider the case where the curves $u =$ const. are great circles and $v =$ const. are their orthogonal trajectories; then $\mathcal{F}$ is zero and $\mathcal{G}$ is a function of v alone By a proper choice of parameters we can have

$$\mathcal{F} = 0, \quad \mathcal{G} = 1. \tag{2}$$

* Application de l'Analyse à la Géométrie, 5me edition, p. 322.

Since $\mathfrak{E}$, $\mathfrak{F}$, $\mathfrak{G}$ cannot be chosen arbitrarily, but must satisfy the Gauss equation*

$$\frac{\partial}{\partial u}\left[\frac{\mathfrak{F}}{\mathfrak{E}\sqrt{\mathfrak{E}\mathfrak{G}-\mathfrak{F}^2}}\frac{\partial\mathfrak{E}}{\partial v}-\frac{1}{\sqrt{\mathfrak{E}\mathfrak{G}-\mathfrak{F}^2}}\frac{\partial\mathfrak{G}}{\partial u}\right]$$
$$+\frac{\partial}{\partial v}\left[\frac{2}{\sqrt{\mathfrak{E}\mathfrak{G}-\mathfrak{F}^2}}\frac{\partial\mathfrak{F}}{\partial u}-\frac{1}{\sqrt{\mathfrak{E}\mathfrak{G}-\mathfrak{F}^2}}\frac{\partial\mathfrak{E}}{\partial v}-\frac{\mathfrak{F}}{\mathfrak{E}\sqrt{\mathfrak{E}\mathfrak{G}-\mathfrak{F}^2}}\frac{\partial\mathfrak{E}}{\partial u}\right]$$
$$=2\sqrt{\mathfrak{E}\mathfrak{G}-\mathfrak{F}^2}, \tag{3}$$

which, for the present case, reduces to

$$\frac{\partial^2\sqrt{\mathfrak{E}}}{\partial v^2}+\sqrt{\mathfrak{E}}=0,$$

we find that $\sqrt{\mathfrak{E}}$ is of the form

$$\sqrt{\mathfrak{E}}=U_1\cos v+U_2\sin v, \tag{4}$$

where U_1, U_2 are arbitrary functions of u alone.

Denote by λ_1, μ_1, ν_1; λ_2, μ_2, ν_2, the direction cosines of the tangents to the curves $v=\text{const.}$, $u=\text{const.}$ respectively on the sphere. They satisfy the following relations:†

$$d\lambda_2=\frac{\partial\sqrt{\mathfrak{E}}}{\partial v}\lambda_1\,du-X\,dv, \quad dX=\sqrt{\mathfrak{E}}\lambda_1\,du+\lambda_2\,dv, \tag{5}$$

and similarly for Y, μ_1, μ_2 and Z, ν_1, ν_2. From the last two we have

$$\frac{\partial\lambda_2}{\partial v}+X=0, \quad \frac{\partial X}{\partial v}=\lambda_2, \tag{6}$$

whence

$$\frac{\partial^2 X}{\partial v^2}+X=0,$$

so that

$$X=U_{11}\cos v+U_{12}\sin v, \tag{7}$$

and in like manner

$$\left.\begin{aligned}Y&=U_{21}\cos v+U_{22}\sin v,\\ Z&=U_{31}\cos v+U_{32}\sin v,\end{aligned}\right\} \tag{7'}$$

where $U_{11}, \ldots, U_{32}$ are functions of u alone. Since $\Sigma X^2=1$, these functions

* Bianchi, Lezioni, p. 67.

† Bianchi, Lezioni, p. 94.

must satisfy the conditions

$$U_{11}^2 + U_{21}^2 + U_{31}^2 = 1, \quad U_{12}^2 + U_{22}^2 + U_{32}^2 = 1, \quad U_{11}U_{12} + U_{21}U_{22} + U_{31}U_{32} = 0. \tag{8}$$

Again from (5) we have

$$\frac{\partial \lambda_2}{\partial u} = \frac{\partial \sqrt{\mathfrak{G}}}{\partial v}\lambda_1, \quad \frac{\partial X}{\partial u} = \sqrt{\mathfrak{G}}\,\lambda_1;$$

eliminating λ_1 and replacing $\sqrt{\mathfrak{G}}$, X, λ_2 by their expressions, we get

$$U_1U_{12}' - U_2U_{11}' = 0, \quad U_1U_{22}' - U_2U_{21}' = 0, \quad U_1U_{32}' - U_2U_{31}' = 0, \tag{9}$$

where the accents denote differentiation with respect to u.

In the first of (1), substitute for $\mathfrak{G}$, X, Y, Z, their expressions from (4), (7); then

$$(U_1 \cos v + U_2 \sin v)^2 = \Sigma\,(U_{11}' \cos v + U_{12}' \sin v)^2,$$

and, consequently,

$$U_1^2 = \Sigma\,U_{11}'^2, \quad U_2^2 = \Sigma\,U_{12}'^2, \quad U_1U_2 = \Sigma\,U_{11}'U_{12}'.$$

Combining these results with (9), we find

$$\left.\begin{aligned} \frac{U_{11}'}{U_{21}U_{32} - U_{31}U_{22}} = \frac{U_{21}'}{U_{31}U_{12} - U_{11}U_{32}} = \frac{U_{31}'}{U_{11}U_{22} - U_{21}U_{12}} = U_1, \\ \frac{U_{12}'}{U_{21}U_{32} - U_{31}U_{22}} = \frac{U_{22}'}{U_{31}U_{12} - U_{11}U_{32}} = \frac{U_{32}'}{U_{11}U_{22} - U_{21}U_{12}} = U_2. \end{aligned}\right\} \tag{10}$$

These conditions, (8), (10), which the six functions $U_{11}, \ldots, U_{32}$ must satisfy in order that the expressions (7) shall be the coordinates of a point on the sphere of radius unity for which the curves $u =$ const. are great circles, are the very conditions which these functions would necessarily satisfy if U_{11}, U_{21}, U_{31} were the direction cosines of the tangent and U_{12}, U_{22}, U_{32} of the binormal of a curve defined by the intrinsic equations

$$\rho = \frac{1}{U_1}, \quad \tau = \frac{1}{U_2},$$

where ρ and τ are the radii of curvature and torsion respectively. Hence, *the*

problem of finding the direction cosines of the lines of a congruence, whose developables are represented on the sphere by a given family of great circles and their orthogonal trajectories, is equivalent to the determination of a skew curve from its intrinsic equations.

When the coordinates of a point on a sphere are given by (7), where $U_{11}, \ldots, U_{32}$ satisfy the conditions (8), the curves $u = \text{const.}$ on the sphere are great circles and $v = \text{const.}$ are their orthogonal trajectories. Hence the inverse of the above problem reduces to the determination of functions satisfying the conditions (8).

2. For the special spherical system which we are considering, the equation of Laplace, found by Guichard* and which is satisfied by the semi-focal distance for any of the congruences with the given representation of their developables, reduces to the form

$$\frac{\partial^2 \rho}{\partial u\, \partial v} + \frac{\partial \log \sqrt{\mathfrak{G}}}{\partial v} \frac{\partial \rho}{\partial u} + \frac{\partial^2}{\partial u\, \partial v} \log \sqrt{\mathfrak{G}} \,.\, \rho = 0. \tag{11}$$

By one quadrature we get

$$\frac{\partial \rho}{\partial v} + \frac{\partial \log \sqrt{\mathfrak{G}}}{\partial v} \rho = V, \tag{11'}$$

where V is a function of v alone, and by a second quadrature

$$\rho = \frac{1}{\sqrt{\mathfrak{G}}} \left(\int \sqrt{\mathfrak{G}}\, V dv + U \right), \tag{12}$$

where U is a function of u alone. Since it can be shown that to every solution of the general equation found by Guichard there corresponds a congruence with the given representation ; it follows that the functions U and V are perfectly arbitrary in the present case.

Guichard has shown that the cartesian coordinates, x_1, y_1, z_1, of a point on the middle surface of the congruences corresponding to a particular solution of equation (11), are given by†

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* Bianchi, Lezioni, p. 262.

† Bianchi, p. 262.

$$\left.\begin{aligned}\frac{\partial x_1}{\partial u} &= \frac{\partial \rho}{\partial u} X - \rho \frac{\partial X}{\partial u}, \\ \frac{\partial x_1}{\partial v} &= -\left(\frac{\partial \rho}{\partial v} + \frac{\partial \log \mathfrak{S}}{\partial v} \rho\right) X + \rho \frac{\partial X}{\partial v},\end{aligned}\right\} \tag{13}$$

and similar expressions in y_1 and z_1.

3. We consider now the congruence corresponding to a given function $\sqrt{\mathfrak{S}}$ and a particular form of ρ, and let S denote one of the surfaces orthogonal to this congruence. Denote by x, y, z the cartesian coordinates of a point on S and by r the distance of the point from the corresponding point on the mean surface; then

$$x = x_1 + rX, \quad y = y_1 + rY, \quad z = z_1 + rZ. \tag{14}$$

Differentiating these equations and multiplying by X, Y, Z respectively, we find

$$dr = -\Sigma X \, dx_1, \tag{15}$$

or

$$\frac{\partial r}{\partial u} = -\Sigma X \frac{\partial x_1}{\partial u}, \quad \frac{\partial r}{\partial v} = -\Sigma X \frac{\partial x_1}{\partial v}. \tag{15'}$$

When the expressions for $\frac{\partial x_1}{\partial u}, \ldots, \frac{\partial z_1}{\partial v}$, as given by (13), are substituted in these equations, they reduce to

$$\frac{\partial r}{\partial u} = -\frac{\partial \rho}{\partial u}, \quad \frac{\partial r}{\partial v} = \frac{\partial \rho}{\partial v} + \frac{\partial \log \mathfrak{S}}{\partial v} \rho. \tag{16}$$

From the first of these we get

$$r = -\rho + V_2,$$

where V_2 is a function of v alone. Substituting this expression in the second of (16), we find that V_2 satisfies the condition

$$V_2' = 2\left(\frac{\partial \rho}{\partial v} + \frac{\partial \log \sqrt{\mathfrak{S}}}{\partial v} \rho\right),$$

where the accent denotes differentiation with respect to v. Comparing this with (11′), we remark that

$$V_2 = 2\int V dv + 2C,$$

where C is a constant, and consequently

$$r = -\rho + 2\int V dv + 2C. \tag{17}$$

From the formulæ (13), (14), (17) we find

$$\left.\begin{aligned} \frac{\partial x}{\partial u} &= 2\left(\int V dv + C - \rho\right)\frac{\partial X}{\partial u}, \\ \frac{\partial x}{\partial v} &= 2\left(\int V dv + C\right)\frac{\partial X}{\partial v}, \end{aligned}\right\} \tag{18}$$

and similarly in y and z.

The preceding development shows that when in these formulæ (18) we assign to ρ a particular form, they give by quadratures the family of parallel surfaces cutting the corresponding congruence orthogonally, each surface of the family being determined by a particular value of the constant C. Since ρ contains two arbitrary functions, U, V, there exists a double infinity of families of parallel surfaces whose lines of curvature in one system are represented on the sphere by a given family of great circles, and after the functions X, Y, Z have been found, the further determination of these surfaces reduces to quadratures.

Write

$$E = \Sigma\left(\frac{\partial x}{\partial u}\right)^2, \quad F = \Sigma\frac{\partial x}{\partial u}\frac{\partial x}{\partial v}, \quad G = \Sigma\left(\frac{\partial x}{\partial v}\right)^2,$$

then from (18) we have

$$E = 4\left[\int V dv + C - \rho\right]^2 \mathfrak{E}, \quad F = 0, \quad G = 4\left[\int V dv + C\right]^2. \tag{19}$$

From the last of these expressions, it follows that G is a function of v alone; hence the lines of curvature whose spherical representation is a family of great circles, are geodesics on the surface, that is, they are plane curves.

4. Recalling the Rodriques formulæ,*

$$\frac{\partial x}{\partial u} = \rho_1 \frac{\partial X}{\partial u}, \quad \frac{\partial x}{\partial v} = \rho_2 \frac{\partial X}{\partial v},$$

where ρ_1 and ρ_2 are the principal radii of curvature of the surface, and comparing them with (18), we get

$$\rho_1 = 2\left(\int V dv + C - \rho\right), \quad \rho_2 = 2\left(\int V dv + C\right). \tag{20}$$

From the second we have that ρ_2 is a function of v alone.

Conversely, let Σ be a surface for which ρ_2 is a function of v alone; we may write

$$\rho_2 = 2\left(\int V dv + C\right).$$

Denote by 2ρ the distance between the focal points and by r the distance from the middle point to the surface; then

$$\rho_2 = (r + \rho) = 2\,(V dv + C).$$

From this we have

$$\frac{\partial r}{\partial u} = -\frac{\partial \rho}{\partial u}, \quad \frac{\partial r}{\partial v} = -\frac{\partial \rho}{\partial v} + 2V,$$

and if x_1, y_1, z_1 are the coordinates of the middle point,

$$\Sigma X \frac{\partial x_1}{\partial u} = \frac{\partial \rho}{\partial u}, \quad \Sigma X \frac{\partial x_1}{\partial v} = \frac{\partial \rho}{\partial v} - 2V.$$

For any normal congruence referred to its developables and with x_1, y_1, z_1 for coordinates of the middle point, we have†

$$\Sigma X \frac{\partial x_1}{\partial u} = \quad \frac{\partial \rho}{\partial u} + \frac{\partial \log \mathcal{G}}{\partial u} \rho,$$

$$\Sigma X \frac{\partial x_1}{\partial v} = -\frac{\partial \rho}{\partial v} - \frac{\partial \log \mathfrak{G}}{\partial v} \rho.$$

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* Bianchi, Lezioni, p. 101.

† Bianchi, Lezioni, p. 262.

Comparing these with the preceding, we have

$$\rho \frac{\partial \log \mathcal{G}}{\partial u} = 0, \quad \frac{\partial \rho}{\partial v} + \rho \frac{\partial \log \sqrt{\mathcal{E}}}{\partial v} = V.$$

These equations are satisfied by $\rho = 0$, $V = 0$. In this case $\rho_1 = \rho_2$, and hence Σ is a sphere. Since every line on a sphere is a line of curvature, it follows that the sphere is an evident solution of our problem. However, when Σ is not a sphere, the first equation shows that $\mathcal{G}$ is a function of v alone, and the second equation which ρ must satisfy is the equation (11'). Hence, *for all surfaces having one of their principal radii constant along each line of curvature in one system, the spherical representation of the other lines of curvature is a family of great circles and these lines are geodesics on the surface.*

5. Consider now the surfaces for which the lines of curvature $u =$ const. are geodesics; then G is a function of v alone. In this case the second Codazzi equation* reduces to

$$\frac{\partial D''}{\partial u} = 0,$$

from which it follows that D'' is a function of v alone. Hence ρ_2, which is equal to $-\frac{D''}{G}$, is a function of v alone, and consequently the surfaces which we are discussing are characterized by the property that their lines of curvature in one system are geodesics.

The first and second sheets of the evolute of any surface may be defined as the envelopes of the planes through a point on the surface and perpendicular to the tangents to the lines of curvature $v =$ const., $u =$ const. through the point. For the surfaces which we are considering, the former plane is the same at all points along a curve $u =$ const. and is, in fact, the plane of the curve. Hence this plane depends entirely upon one parameter u, and, consequently, its envelope is a developable surface. Conversely, when the first sheet is a developable, the curves $u =$ const., on the surface are geodesics. From this it is seen that the surfaces which we have been considering are the very ones which Monge dis-

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* Ib., p. 91.

cussed as surfaces for which one of the focal sheets is a developable surface and which are known as the *surfaces of Monge.** Hence, *surfaces whose lines of curvature in one system are represented on the sphere by great circles are surfaces of Monge, and conversely.*

6. Monge has shown that the most general surfaces of this kind are generated by an invariable plane curve whose plane rolls without sliding upon a developable surface. The radius of curvature of the curve is, in this case, the radius of principal curvature of the surface corresponding to the lines of curvature which are the successive positions of the plane curve, and consequently the latter radius depends only upon the parameter of these lines of curvature; in the preceding development this radius was denoted by ρ_2. We have found for its expression,

$$\rho_2 = 2\int V\,dv + 2C. \tag{21}$$

Since v is the parameter of the curvature, it follows from this expression that for a definite form of the function V the character of the curve is completely determined and the variation of C gives parallel curves with the same evolute. Recalling the expression for the semi-focal distance,

$$\rho = \frac{1}{\sqrt{\mathfrak{G}}}\left(\int \sqrt{\mathfrak{G}}\, V\,dv + U\right), \tag{22}$$

we remark that the arbitrary function V, which appears in this expression, is the same as in (21), and consequently when a particular form is given to the function V in (22), the character of the geodesic lines of curvature of the surface is determined, which are to be represented on the sphere by a given family of great circles. We will now find in what way the function U serves to determine the surface.

Since the planes of the great circles on the sphere are parallel to the planes enveloping the first sheet of the evolute, the intersections of the former are parallel to the lines generating the developable. Hence, when the spherical representation of the surface is given, the directions of the generatrices of the

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* Monge, "Application de l'Analyse à la Géométrie, 5me edition, p. 322 et seq.

developable are determined, and consequently the function U determines the manner in which these lines are arranged so as to form the developable. For example, if all the great circles on the sphere have a common diameter, all the planes of these circles intersect in this line, and consequently the generating developable of the surface is cylindrical. In this case the function U determines the character of the right section of the cylinder. The corresponding surfaces are the so-called *moulure* surfaces. We consider this special case further and show in what manner U enters into the expressions for the cartesian coordinates of the surface S.

Let the axis of z be the common diameter of all the great circles, and therefore parallel to the axis of the cylinder. The coordinates of a point on the sphere can be written

$$X = \cos u \sin v, \quad Y = \sin u \sin v, \quad Z = \cos v,$$

and from this

$$\sqrt{\mathfrak{G}} = \sin v, \quad \mathcal{G} = 1.$$

Now (22) becomes

$$\rho = \frac{1}{\sin v}\left(\int \sin v \, V dv + U\right),$$

and from (13) we have, by a quadrature, for the coordinates of the middle surface of the congruence,

$$x_1 = \int (U' \cos u + U \sin u)\, du - \cos u \int V \sin v \, dv,$$

$$y_1 = \int (U' \sin u - U \cos u)\, du - \sin v \int V \sin v \, dv,$$

$$z_1 = U \cot v - \int \frac{V \cos v \sin^2 v + \int V \sin v \, dv}{\sin^2 v} dv.$$

If we denote by x, y, z the coordinates of the surface S corresponding to the value zero for C in (17), and substitute the value for r and the preceding expressions for x_1, y_1, z_1 in (14), the surface S is given by

$$\left.\begin{aligned} x &= \quad 2\int U \sin u\, du + 2\cos u \int V_1 \cos v\, dv\,, \\ y &= -2\int U \cos u\, du + 2\sin u \int V_1 \cos v\, dv\,, \\ z &= -2\int V_1 \sin v\, dv\,, \end{aligned}\right\} \tag{23}$$

where

$$V_1 = \int V dv\,.$$

From the above we find for the square of the linear element of S,

$$ds^2 = 4\left(U - \int V_1 \cos v\, dv\right)^2 du^2 + V_1^2\, dv^2. \tag{24}$$

It is evident that the surfaces of revolution are a particular class of the surfaces defined by (23) and correspond to the case where the cylinder reduces to its axis. From (24) it follows that for surfaces of revolution U is a constant, and conversely.

7. Since the evolute of a line of curvature $u = \text{const.}$ is a plane curve in the same plane as the latter, and since the locus of these evolutes is the second sheet S_2 of the evolute of S, this second sheet also is a surface of Monge. Since its generating plane is the same as for S, the family of great circles on the sphere is the same for both surfaces. However, the normals to these two surfaces at corresponding points are perpendicular to one another. Consequently if the point (u, v) on the sphere is the image of a point on S, then $\left(u,\, v + \frac{\pi}{2}\right)$ is the image of the corresponding point on S_2. Again, since the differential dv denotes the angle between consecutive radii of the sphere along a great circle, or, what is the same thing, the angle between consecutive tangents to the geodesics $u = \text{const.}$ of the surface, the evolute of the curve

$$\rho_2 = 2V_1$$

is given by

$$\rho_2' = 2\frac{dV_1}{dv}\,.$$

As the generating developable is the same for both S and S_2, the function U is

the same for both. Therefore, given a surface S corresponding to a system of expressions for the functions $\sqrt{\mathfrak{G}}$, U and V_1; if, in the first $\cos v$, $\sin v$, are replaced by $-\sin v$, $\cos v$ respectively and V_1 is replaced by $\frac{dV_1}{dv}$, the corresponding surface is the second sheet of the envelope of S.

For example, if we make these changes in (23), we have the following expressions for the coordinates of the corresponding surface S_2:

$$\left.\begin{aligned}
x_2 &= \quad 2\int U \sin u\, du - 2\cos u \int \frac{dV_1}{dv} \sin v\, dv, \\
y_2 &= -\,2\int U \cos u\, du - 2\sin u \int \frac{dV_1}{dv} \sin v\, dv, \\
z_2 &= -\,2\int \frac{dV_1}{dv} \cos v\, dv.
\end{aligned}\right\} \tag{25}$$

From the definition of the evolute, we have that

$$x_2 = x - \rho_2 X, \quad y_2 = y - \rho_2 Y, \quad z_2 = y - \rho_2 Z.$$

If the previously found expressions for x, y, z, ρ_2, X, Y, Z are substituted here, these equations can be brought to the form (25).

8. We have remarked that surfaces of revolution belong to the class under discussion and have given an example of a spherical representation of these surfaces. Now we wish to find all possible forms of the functions U_1, U_2, U corresponding to surfaces of revolution. We do this by expressing the condition that ρ_1 shall be a function of v alone. From (12) this gives

$$\frac{U_1\int V \cos v\, dv + U_2\int V \sin v\, dv + U}{U_1 \cos v + U_2 \sin v} = \Phi(v).$$

The following cases give all the possible ways in which this equation of condition is satisfied, and, furthermore, there are surfaces of revolution corresponding to each of these cases.

1°. $U_2 = 0, \quad U = \lambda U_1,$

where λ is a constant equal to or different from zero. From (10) we find that

U_{12}, U_{22}, U_{32} are constants ; put

$$U_{12} = c_1, \quad U_{22} = c_2, \quad U_{32} = c_3.$$

On account of the relation between $U_{11}, \ldots, U_{32}$ we can introduce two functions θ_1 and θ_2 as follows:

$$U_{11} = \sin\theta_1 \cos\theta_2, \quad U_{21} c_3 - U_{31} c_2 = \sin\theta_1 \sin\theta_2, \quad c_1 = \cos\theta_1.$$

From (10)

$$\theta_2' = U_1,$$

whence

$$\theta_2 = \gamma_1 + \int U_1\, du,$$

where γ_1 is a constant. Therefore

$$U_{11} = \sqrt{1 - c_1^2} \cos\left(\gamma_1 + \int U_1\, du\right),$$

and similarly for U_{21} and U_{31}. Hence X, Y, Z are given by quadratures and, therefore, the complete determination of the surfaces of revolution, for which $\sqrt{\mathfrak{G}} = U_1 \cos v$, reduces to quadratures. This is the case previously discussed at which time we took $U_1 = 1$.

2°. $U_1 = 0$, $U = \lambda U_2$.

This case is similar to the preceding and leads to similar results.

3°. $U_1 = \lambda U_2 = \mu U$,

where λ, μ are constants, and U is either a function of u or a constant different from zero. Recalling the preceding results and remarking that $\frac{U_1}{U_2} = \lambda$, we see that the problem of determining the direction cosines of the normals to the surface is equivalent to that of finding general helices from their intrinsic equations. By methods similar to those used in case 1°, it can be shown that this determination reduces to quadratures. Hence the surfaces of revolution, for which $\sqrt{\mathfrak{G}} = U_1(\cos v + \lambda \sin v)$, are found by quadratures.

4°. $U_1 = \lambda U_2$, $U = \mu$,

where λ, μ are constants. This case is similar to the preceding.

9. Since ρ_2 is a function of v alone, for ρ_1 to be a function of ρ_2 it also would

be a function of v alone, and, consequently, S would be a surface of revolution. Hence, *surfaces of revolution are the only surfaces of Monge which are Weingarten surfaces.*

Again, from (19),

$$\sqrt{\frac{E}{G}} = \frac{\left(\int V dv - \rho\right)\sqrt{\mathfrak{G}}}{\int V dv} = \frac{U_2 \int V_1 \cos v \, dv - U_1 \int V_1 \sin v \, dv - U}{V_1}.$$

For S to be an isothermic surface, it is necessary and sufficient that the numerator of the right-hand member be a product of a function of u by a function of v. It is readily seen that the four cases 1°, 2°, 3°, 4° of the preceding section are the only ones satisfying this condition; hence, *the surfaces of revolution are the only surfaces of Monge which are isothermic.*

10. It is well known that the direction cosines X, Y, Z, corresponding to a given family of great circles and their orthogonal trajectories, are particular solutions of the equation*

$$\frac{\partial^2 \theta}{\partial u \, \partial v} - \frac{\partial \log \sqrt{\mathfrak{G}}}{\partial v} \frac{\partial \theta}{\partial u} = 0, \tag{26}$$

and the envelope of the plane whose equation is

$$Xx + Yy + Zz = W,$$

where W is a particular solution of the above equation, has the curves $u =$ const., $v =$ const. for lines of curvature. Moreover, the cartesian coordinates of the point of contact are†

$$x = WX + \Delta(W, X), \quad y = WY + \Delta(W, Y), \quad z = WZ + \Delta(W, Z),$$

where $\Delta(\phi, \psi)$ is the mixed differential parameter defined by

$$\Delta(\phi, \psi) = \frac{1}{\mathfrak{E}\mathfrak{G} - \mathfrak{F}^2}\left\{\mathfrak{G}\frac{\partial \phi}{\partial u}\frac{\partial \psi}{\partial u} - \mathfrak{F}\left(\frac{\partial \phi}{\partial u}\frac{\partial \psi}{\partial v} + \frac{\partial \phi}{\partial v}\frac{\partial \psi}{\partial u}\right) + \mathfrak{E}\frac{\partial \phi}{\partial v}\frac{\partial \psi}{\partial v}\right\}.$$

The general integral of (26) is readily found to be

$$W = \cos v \int U U_1 du + \sin v \int U U_2 du + V, \tag{27}$$

*** Bianchi, Lezioni, p. 119.** **† Ib., p. 137.**

where U, V are functions of u and v respectively. From (7) we see that the problem of finding X, Y, Z reduces to taking zero for V and the choice of three sets of values for U so that the resulting forms shall satisfy conditions (8) and (10). After X, Y, Z have been found and the two quadratures in (27) for any value of U have been effected, the corresponding surface S is found by differentiation. When we compare this result with those of the preceding sections, we see that the actual determination of the surfaces requires fewer quadratures, but an inspection of the above formulæ shows that the preceding method leads to a more ready geometrical interpretation of the results.

It is of interest to remark that when $V = 0$ in (27), W has the same form as $\sqrt{\mathfrak{E}}$. Substituting $\sqrt{\mathfrak{E}}$ for θ in (26), we find $U_1 = \lambda U_2$, where λ is a constant, taking any value. Hence, when

$$\sqrt{\mathfrak{E}} = U_1(\lambda \cos v + \mu \sin v), \quad \mathcal{F} = 0, \quad \mathcal{G} = 1,$$

the envelope of the plane

$$Xx + Yy + Zz = \sqrt{\mathfrak{E}}$$

is a surface of revolution having the lines $u =$ const. for meridians and $v =$ const. for parallels.

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